REPRESENTATION OF VECTOR VALUED
NONLINEAR FUNCTIONS

BY

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Abstract. A representation theorem for "additive" nonlinear functionals on spaces $L^p(\mu)$ is here extended to "additive" nonlinear functions from Lebesgue-Bochner function spaces $L_E^p(\mu)$ (E a separable Banach space) into Banach spaces B. A counterexample is provided to show that the restriction to separable E is essential.

Let (X, Σ, μ) be a measure space, B be an arbitrary Banach space, and E be a separable Banach space. Let M_E be the vector space of E-valued Bochner measurable functions on X. A function $F: M_E \to B$ is said to be *additive* if $F(x+y) = F(x) + F(y)$ whenever $x, y \in M_E$ with (almost everywhere) disjoint supports. More specifically we are concerned with continuous additive functions defined on the associated L_E^p spaces ($1 \leq p \leq \infty$) of Bochner μ-integrable functions. The main problem that arises here is to characterize the additive functions on $L_E^p(\mu) \to B$ which admit an integral representation of the form $F(x) = \int_X \varphi(x(\xi), \xi)\, d\mu(\xi)$ where φ is required to satisfy certain conditions related to those occurring in the theory of nonlinear integral equations, Krasnosel'skiǐ [5]. Several authors considered this problem when both E and B are the real line, R. See for these and related results Mizel [6], Mizel and Sundaresan [7], Drewnowski and Orlicz [2] and Woyczyński [11]. It will be seen that the case $E = R$, $p = \infty$ is the crucial one. Once the results are available for this case then, by straightforward techniques, one can handle the general case of $F: L_E^p(\mu) \to B$ ($1 \leq p \leq \infty$).

The present study is motivated by the following considerations:

1. The representation theorems in [6] and [2] are false in the B-valued case even for $E = R$. We furnish an example in §3 justifying this assertion.

2. The results in this paper are applicable to investigations based on Lebesgue (pseudo) differentiability of vector valued additive set functions as well as to investigations based on Radon-Nikodym differentiability of such functions. They also provide a sufficient condition that certain types of vector valued random processes have continuous sample paths.

3. Integral representations of the type considered here have, for the case of functionals (i.e. $B = R$), already been studied extensively when the functionals are

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linear (Edwards [4, Chapter 8]), and our methods provide an alternate proof for certain of the latter representation theorems (see §4).

The techniques that are adopted in the proofs of various assertions are very similar to those in [6], [7], and [2], apart from the use of the general Radon-Nikodym theorem established in a recent paper of Rieffel [10], and of a theorem of Phillips on Lebesgue differentiation.

The paper is divided into four sections. In §1 after recalling the notations and definitions we proceed to establish the representation theorem for B-valued additive functions on $L_R^\infty(\mu)$ and to deduce from this theorem the representation theorems for the case of $L_R^p(\mu)$, $1 \leq p < \infty$. In §2 we introduce the Carathéodory differential of a 1-parameter family of additive set functions and obtain a necessary and sufficient condition for the differentiability of such a family in this sense. In §3 we provide an example illustrating that the theorems in [6] and [2] are not true in the B-valued case even when B is a separable Hilbert space and $E = R$. In §4 we point out how the results of §1 can be extended to the case of an arbitrary separable Banach space E.

1. **Representation theorems.** In the sequel (X, Σ, μ) is a complete σ-finite positive measure space unless otherwise specified and B is an arbitrary Banach space. R is the real line. $L^\infty(\mu)$ is the space of equivalence classes of essentially bounded (real-valued) measurable functions x, with the norm $\|x\|$ $= \text{ess sup} \{|x(\xi)| \mid \xi \in X\}$. $\mathscr{L}_B^p$ is the collection of Bochner p-summable B-valued functions on X and L_B^p $(1 \leq p < \infty)$ is the corresponding Banach space of equivalence classes of p-summable functions.

A function $\varphi: R \times X \to B$ is said to be a *Carathéodory* function on X to B, in short $\varphi \in \text{Car}(X; B)$ (Car B when the measure space is understood), if $\varphi(c, \cdot)$ is a Bochner measurable function for each real number c and $\varphi(\cdot, \xi)$ is a continuous function for all ξ outside a μ-null set. A function $F: L^\infty(\mu) \to B$ is called *additive* ("orthogonally additive" according to [2]) if $F(x+y) = F(x) + F(y)$ for $x, y \in L^\infty(\mu)$ with disjoint support, i.e. $xy = 0$ μ.a.e. Given a finitely additive set function $G: \Sigma \to B$ having finite variation, the variation of G on a set $E \in \Sigma$ is denoted by $V(G)(E)$. If $\{G_t\}_{t \in R}$ is a one parameter family of finitely additive set functions on Σ and d, δ are two positive numbers then $V_d(\delta, G)$ denotes the quantity

$$V_d(\delta, G) = \sup \sum_{i=1}^{n} V(G_{h_1^i} - G_{h_2^i})(E_i)$$

where the supremum is taken over all ordered n-tuples, $n \geq 1$, of pairwise disjoint measurable sets $\{E_1, \ldots, E_n\}$ with $h_1^i, h_2^i \in [-d, d]$ and $|h_1^i - h_2^i| \leq \delta$, $1 \leq i \leq n$. The family $\{G_t\}$ is said to be *locally uniformly continuous in variation* provided $\lim_{\delta \to 0+} V_d(\delta, G) = 0$ for every $d > 0$. Finally, if F is an additive function on $L^\infty(\mu)$ and h is a real number then F_h will denote the finitely additive set function on $\Sigma \to B$ defined by $F_h(E) = F(hI_E)$ where I_E is the characteristic function of E.

We recall certain concepts concerning vector valued set functions studied in detail in [10]. The *average range* of an additive set function $G: \Sigma \to B$ on a measurable set E, $0 < \mu(E) < \infty$, is the set

$$A_E(G) = \{G(E')/\mu(E') \mid E' \in \Sigma, E' \subset E, 0 < \mu(E')\}.$$

A set function G has *locally almost compact* average range if whenever $E \in \Sigma$, $0 < \mu(E) < \infty$, and $\varepsilon > 0$ then there exists a measurable set $E_0 \subset E$ such that $\mu(E - E_0) < \varepsilon$ and $A_{E_0}(G)$ is a precompact set in B. (See Proposition 1.12 on p. 472 in [10].)

Before proceeding to the representation of the additive functions we state several useful lemmas.

LEMMA 1. *If $F: L^\infty(\mu) \to B$ is an additive function such that $\{F_t\}_{t \in R}$ is locally uniformly continuous in variation then F is uniformly continuous on bounded subsets of simple functions.*

Proof. Let S be the subspace of $L^\infty(\mu)$ consisting of simple functions. Let $d > 0$ be a fixed positive number and $S_d = \{x \mid x \in S, \|x\|_\infty \leq d\}$. The hypothesis on F in the lemma implies that for each $\varepsilon > 0$ there exists a positive number $\delta(\varepsilon)$ such that if $\delta \leq \delta(\varepsilon)$ then $V_d(\delta, F) \leq \varepsilon$. Let now $x, y \in S_d$ be such that $\|x - y\|_\infty \leq \delta(\varepsilon)$. Thus there exists a finite collection of pairwise disjoint measurable sets $\{E_i\}_{1 \leq i \leq n}$ such that $x = \sum a_i I_{E_i}$, $y = \sum b_i I_{E_i}$ with $|a_i - b_i| \leq \delta(\varepsilon)$. Thus

$$\|F(x) - F(y)\| \leq \sum_{i=1}^n \|(F_{a_i} - F_{b_i})(E_i)\| \leq \sum_{i=1}^n V(F_{a_i} - F_{b_i})(E_i) < \varepsilon.$$

REMARK 1. (a) The converse of the assertion in the preceding lemma is false, i.e. the uniform continuity of F on bounded subsets of S need not imply that $\{F_t\}_{t \in R}$ is locally uniformly continuous in variation. This is discussed at the end of the example in §3.

(b) For convenience of reference we note that if $f: X \to B$ is a function in L_B^1 and if, for each $E \in \Sigma$, $G(E) = \int_E f \, d\mu$ then $V(G)(E) = \int_E \|f(\xi)\| \, d\mu(\xi)$. For a proof of this fact we refer to Dunford and Schwartz [3].

LEMMA 2. *If $\varphi: R \times X \to B$ is a Carathéodory function on X to B and $T \in \Sigma$ has finite measure then corresponding to any three preassigned positive numbers δ, ε and A there exists a measurable set M, and a positive number η such that (1) $\mu(M) \leq \delta$ and (2) for all $\xi \in T \sim M$, $\|\varphi(t, \xi) - \varphi(t', \xi)\| \leq \varepsilon$ if $|t - t'| \leq \eta$, $|t|, |t'| \leq A$.*

Proof. Let us define a sequence of measurable sets $\{X_n\}_{n \geq 1}$ as follows:

$$X_n = \{\xi \mid t, t' \in [-A, A], |t - t'| \leq 1/n \Rightarrow \|\varphi(t, \xi) - \varphi(t', \xi)\| \leq \varepsilon\}.$$

If $\{(a_n^i, b_n^i)\}_{i \geq 1}$ is an enumeration of all rational number pairs in $[-A, A]$ such that $|a_n^i - b_n^i| \leq 1/n$ then

$$X_n = \bigcap_{i \geq 1} \{\xi \mid \|\varphi(a_n^i, \xi) - \varphi(b_n^i, \xi)\| \leq \varepsilon\} - N_n,$$

where N_n is a subset of the null set in X for which $\varphi(\cdot, \xi)$ is not continuous. Thus X_n is a measurable set for all $n \geq 1$. Since $\varphi(\cdot, \xi)$ is uniformly continuous on $[-A, A]$ for almost all ξ it follows that $\mu(T \sim \bigcup_{n \geq 1} X_n) = 0$. Since $X_n \uparrow T$ there exists an integer N such that $\mu(T \sim X_N) \leq \delta$ and, for all $\xi \in X_N$, $\|\varphi(t, \xi) - \varphi(t', \xi)\| \leq \varepsilon$ if $t, t' \in [-A, A]$ and $|t - t'| \leq 1/N$. Choosing for M and η, X_N and $1/N$ respectively, the proof is completed.

REMARK 2. We note that the dependence of M on the given ε is inessential. Actually the set M could be chosen independent of ε and A, by utilizing sequences $\{A_n\}, \{\varepsilon_n\}, \{\delta_n\}$ with $A_n \to \infty$, $\varepsilon_n \to 0$, and $\sum \delta_n < \delta$.

This is easily seen from the proof in the next lemma.

LEMMA 3. *Let $f: R \to \mathscr{L}_B^1$ where $\mathscr{L}_B^1$ is the space of all B-valued Bochner integrable functions on X and for each $t \in R$ let F_t denote the indefinite integral of $f(t)$. Then if the family $\{F_t\}_{t \in R}$ is locally uniformly continuous in variation there exists a function $\varphi \in \mathrm{Car}\, B$ such that, for all t, $f(t)(\xi) = \varphi(t, \xi)$ almost everywhere.*

Proof. As a first step we verify that, for each positive number d, $f_t(\xi) = f(t, \xi)$ is uniformly continuous on the set of rationals $Q_d \subset [-d, d]$ at all ξ outside a μ-null set. Let $\varepsilon > 0$ and let

$$A_n = \{\xi \mid \text{there exist } t_1, t_2 \in Q_d \text{ with } |t_1 - t_2| \leq 1/n \text{ and } \|f(t_1, \xi) - f(t_2, \xi)\| \geq \varepsilon\}.$$

It is verified that $\{A_n\}$ is a nested family of measurable sets. Further $\lim \mu(A_n) = 0$ as $n \to \infty$. For if $\lim \mu(A_n) \neq 0$ there exists a positive number a such that $\mu(A_n) \geq a$. Given $\delta > 0$ let n be chosen so that $1/n \leq \delta$. Let $\{(p^i, q^i)\}_{i \geq 1}$ be an enumeration of all rational pairs in $Q_d \times Q_d$ such that $|p^i - q^i| \leq 1/n$ and let

$$A_n^i = \{\xi \mid \|f(p^i, \xi) - f(q^i, \xi)\| \geq \varepsilon\}.$$

Let $C_n^{i+1} = A_n^{i+1} \sim \bigcup_{j \leq i} A_n^j$. Thus $a \leq \mu(A_n) = \sum_{i=1}^{\infty} \mu(C_n^i)$. Let N be so chosen that $\sum_{i=1}^{N} \mu(C_n^i) \geq a/2$. Now $\sum_{i=1}^{N} \int_{C_n^i} \|f(p^i, \xi) - f(q^i, \xi)\| d\mu(\xi) \geq a\varepsilon/2$. Hence by Remark 1(b) it follows that $\lim_{\delta \to 0+} V_d(\delta, F) \neq 0$. That is the family $\{F_t\}_{t \in R}$ is not locally uniformly continuous in variation contrary to the hypothesis. Thus if $A^\varepsilon = \bigcap_{n \geq 1} A_n$ then $\mu(A^\varepsilon) = 0$. Now let $B_d = \bigcup_{m \geq 1} A^{1/m}$. Then $\mu(B_d) = 0$ and if $\xi \notin B_d$, $f(\cdot, \xi)$ is uniformly continuous on the set Q_d. Let us extend the function $f(\cdot, \xi)$ at each $\xi \notin B_d$ to a function $\varphi_d(\cdot, \xi)$ where $\varphi_d(\cdot, \xi)$ is a continuous function on the interval $[-d, d]$. Such a function is uniquely determined by $f(\cdot, \xi)$ since Q_d is dense in $[-d, d]$. Let now $\varphi(\cdot, \xi) = \bigcup_{n \geq 1} \varphi_n(\cdot, \xi)$ for $\xi \in X \sim \bigcup_{n \geq 1} B_n$, and $\varphi(\cdot, \xi) = 0$ for $\xi \in \bigcup_{n \geq 1} B_n$. Since $\mu(\bigcup_{n \geq 1} B_n) = 0$ it follows that for all rational numbers t and for $\xi \notin \bigcup_{n \geq 1} B_n$, $f(t)(\xi) = \varphi(t, \xi)$.

From this it follows that $F_t(E) = \int_E \varphi(t, \xi) d\mu(\xi)$ for all $E \in \Sigma$ and $t \in R$. For let $t \in R$ and let t_n be a sequence of rationals such that $t_n \to t$. From the local uniform continuity in variation of $\{F_t\}_{t \in R}$ it follows that (see Remark 1(b))

$$\int_E \|f(t_n, \xi) - f(t, \xi)\| d\mu(\xi) \to 0.$$

Thus the functions $f(t_n, \cdot)$, and therefore the functions $\varphi(t_n, \cdot)$ are uniformly bounded in L_B^1 norm on E and in addition have the property that their indefinite integrals are uniformly absolutely continuous (Theorem 6, p. 122 in [3]), i.e. $\int_U \|f(t_n)(\xi)\| \, d\mu(\xi) \to 0$ for $U \subset E$ as $\mu(U) \to 0$, uniformly in n. In addition, the functions $\varphi(t_n, \cdot)$ satisfy $\varphi(t_n, \xi) \to \varphi(t, \xi)$ a.e. Thus by Vitalli's convergence theorem [3, p. 150], equicontinuity being clear,

$$F_t(E) = \int_E f(t, \xi) \, d\mu(\xi) = \lim \int_E f(t_n, \xi) \, d\mu(\xi)$$

$$= \lim \int_E \varphi(t_n, \xi) \, d\mu(\xi) = \int_E \varphi(t, \xi) \, d\mu(\xi).$$

Since this holds for all $E \in \Sigma$, $\varphi(t, \xi) = f(t, \xi)$ a.e. which completes the proof.

REMARK 3. Lemma 3 has an obvious interpretation in probability theory. It provides a sufficient condition that a B-valued random function ("stochastic process") f on the real axis, with values $f(t)$ in $\mathscr{L}_B^1(X)$, be equivalent to ("have a version as") a B-valued random function φ possessing almost surely continuous sample paths. On the other hand this condition is not necessary, as is seen by the following example with $B = R$ and with $X =$ the interval $0 \le \xi \le 1$ with Lebesgue measure. We take

$$f(t, \xi) = \varphi(t, \xi) = t^{-1}, \qquad\qquad |\xi - 2t| < t,$$
$$= |\xi - 2t|^{-1} k(t, \xi), \qquad |\xi - 2t| \ge t,$$

where k is a smooth function on $R \times [0, 1] - \{(0, 0)\}$ which is identically 1 for $|\xi - 2t| \le 3/2t$ and identically 0 for $|\xi - 2t| \ge 2t$. Then with $h_1^1 = \delta/3$, $h_2^1 = 0$ and $E_1 = (2\delta/3, \delta)$ one deduces that $V_d(\delta, F) \ge 1$ for all δ. (Conditions of a different type are discussed in [15, Chapter III].)

We next proceed to the key representation theorem of the paper. We state as separate theorems the necessary and sufficient conditions guaranteeing the representation.

THEOREM 1. *Let F be an additive function on $L^\infty(\mu)$ into a Banach space B such that*

(1) *$x_n \to x$ boundedly a.e. $\Rightarrow \|F(x_n) - F(x)\| \to 0$,*

(2) *for each real number t, F_t has almost locally compact average range,*

(3) *for each real number t, $V(F_t)$ is a finite measure,*

(4) *the family $\{F_t\}_{t \in R}$ is locally uniformly continuous in variation.*

Then there exists a function $\varphi \in \mathrm{Car}\ B$ such that for all $x \in L^\infty(\mu)$ the function $\varphi \circ x(\xi) = \varphi(x(\xi), \xi)$ is in L_B^1, and

$$F(x) = \int_X \varphi(x(\xi), \xi) \, d\mu(\xi).$$

Further such a representing function φ can be chosen to satisfy $\varphi(0, \xi) = 0$ for ξ a.e. and is then unique up to sets of the form $R \times N$ where N is a μ-null set.

Proof. Condition (1) implies that for each real number t the set function F_t is countably additive. Thus F_t is a B-valued measure. By the "main theorem" and the remarks on p. 479 in [10] our conditions (2) and (3) imply that for every real number t there exists a function $f_t \in \mathscr{L}_B^1$ uniquely defined up to null sets such that $F_t(E) = \int_E f_t(\xi) \, d\mu(\xi)$ for all $E \in \Sigma$. Since the family of set functions $\{F_t\}_{t \in R}$ satisfy the conditions in the preceding lemma it follows that there exists a function $\varphi \in \operatorname{Car} B$ such that for all real numbers t and for all sets $E \in \Sigma$

$$F_t(E) = F(t I_E) = \int_E \varphi(t, \xi) \, d\mu(\xi).$$

Thus by additivity of F, if x is a simple function, $F(x) = \int_X \varphi(x(\xi), \xi) \, d\mu(\xi)$. Let now $x \in L^\infty(\mu)$. Then there exists a sequence $\{x_n\}_{n \geq 1}$ of simple functions in $L^\infty(\mu)$ such that $\|x_n - x\|_\infty \to 0$ and $x_n(\xi) \to x(\xi)$ a.e. Since $\varphi \in \operatorname{Car} B$ it is verified that $\varphi(x_n(\xi), \xi) \to \varphi(x(\xi), \xi)$ a.e. Further, condition (4) on F together with the fact that $\{x_n\}_{n \geq 1}$ is a Cauchy sequence in $L^\infty(\mu)$ implies that $\{\varphi(x_n(\xi), \xi)\}_{n \geq 1}$ is a Cauchy sequence in L_B^1. Hence it follows that $\int_X \varphi(x_n(\xi), \xi) \, d\mu(\xi) \to \int_X \varphi(x(\xi), \xi) \, d\mu(\xi)$. Thus by condition (1) $F(x) = \int_X \varphi(x(\xi), \xi) \, d\mu(\xi)$. Note that by the construction $\varphi(0, \xi) = 0$ a.e.

For the proof of the uniqueness assertion in the theorem observe that for any $\bar\varphi \in \operatorname{Car} B$ giving a representation of F similar to that provided by the above function φ, one has

$$F_t(E) = F(t I_E) = \int_X \bar\varphi(t I_E(\xi), \xi) \, d\mu(\xi)$$

$$= \int_E (\bar\varphi(t, \xi) - \bar\varphi(0, \xi)) \, d\mu(\xi) + \int_X \bar\varphi(0, \xi) \, d\mu(\xi).$$

Thus, since $F(0) = 0$ it is necessary that (1) $\int_X \bar\varphi(0, \xi) \, d\mu(\xi) = 0$, and (2) for each $t \in R$, $\bar\varphi(t, \cdot) - \bar\varphi(0, \cdot)$ is a Radon-Nikodym derivative of F_t. The requirement that $\bar\varphi(0, \cdot) = 0$ a.e. thus implies that $\bar\varphi(t, \xi) = \varphi(t, \xi)$ for each rational t and each $\xi \notin N_0$ where N_0 is a null set. It then follows by definition of $\operatorname{Car} B$ that $\bar\varphi = \varphi$ except on a set $R \times N$, as asserted.

We proceed next to establish the converse of the preceding theorem.

THEOREM 2. *Let $\varphi \in \operatorname{Car} B$ be such that* (1) *$\varphi(0, \xi) = 0$ for ξ a.e. and* (2) *the function $\varphi \circ x(\xi) = \varphi(x(\xi), \xi)$ is in L_B^1 for all $x \in L^\infty(\mu)$. If for $x \in L^\infty(\mu)$ one defines*

$$F(x) = \int_X \varphi(x(\xi), \xi) \, d\mu(\xi)$$

then F is an additive function satisfying the conditions (1) *to* (4) *of the preceding theorem.*

Proof. Let $\{x_n\}_{n \geq 1}$ be a sequence of functions in $L^\infty(\mu)$ such that $x_n(\xi) \to x(\xi)$ boundedly a.e. Note that if $\psi(t, \xi) = \|\varphi(t, \xi)\|$ then $\psi \in \operatorname{Car} R$ and the function

$x \to \psi \circ x$ maps $L^\infty(\mu) \to L^1(\mu)$. Thus by the proof of Theorem 1 in Mizel [6] it follows that

$$\|\varphi\| \circ x_n = \psi \circ x_n \to \psi \circ x = \|\varphi\| \circ x \quad \text{in } L^1(\mu) \text{ norm.}$$

Hence the indefinite integrals $\int \varphi(x_n(\xi), \xi) \, d\mu(\xi)$ are uniformly absolutely continuous. Moreover $\varphi \circ x_n \to \varphi \circ x$ a.e. Thus by Vitalli's theorem (Dunford-Schwartz [3]) $\|F(x_n) - F(x)\| \to 0$. By the main theorem in [10] it follows that F satisfies conditions (2) and (3). We proceed to verify (4). Using Remark 1(b) it can be seen that it is enough to verify that the mapping $x \to \varphi \circ x$ of $L^\infty(\mu)$ into L^1_B is uniformly continuous on bounded subsets of simple functions in $L^\infty(\mu)$. Assume now the following: (A) there exist positive numbers d and a and two sequences of simple functions $\{x_n\}$, $\{y_n\}$ such that $\|x_n - y_n\|_\infty \le 1/n$, $\|x_n\|_\infty$, $\|y_n\|_\infty \le d$, but $\int_X \|\varphi \circ x_n - \varphi \circ y_n\| \, d\mu \ge a$ for all n. Let us consider first the case in which (X, Σ, μ) is a finite measure space.

Let $\delta_1 > 0$ be so chosen that if $E \in \Sigma$, $\mu(E) < 2\delta_1$, then $\int_E \|\varphi \circ x_1 - \varphi \circ y_1\| \, d\mu \le a/3$. Such a δ_1 certainly exists since the indefinite integral $\int \|\varphi \circ x_1 - \varphi \circ y_1\| \, d\mu$ is μ-absolutely continuous. Corresponding to the pair of real numbers $(\delta_1, a/3\mu(X))$ it follows from Lemma 2 that there exists a measurable set S_1 and an integer n_2 such that $\mu(X - S_1) \le \delta_1$ and for $\xi \in S_1$, $\|\varphi(t, \xi) - \varphi(t', \xi)\| \le a/3\mu(X)$ if $|t - t'| \le 1/n_2$, $|t|, |t'| \le d$. Hence by the choice of $\{x_n\}$, $\{y_n\}$ it follows that $\int_{X-S_1} \|\varphi \circ x_{n_2} - \varphi \circ y_{n_2}\| \, d\mu \ge 2a/3$ and $\int_{X-S_1} \|\varphi \circ x_1 - \varphi \circ y_1\| \, d\mu \le a/3$. Thus if $S_2 = X - S_1$ then $S_1, S_2, n_1 = 1$ and n_2 satisfy

$$\int_{S_1} \|\varphi \circ x_{n_1} - \varphi \circ y_{n_1}\| \ge \tfrac{2}{3}a, \qquad \int_{S_2} \|\varphi \circ x_{n_1} - \varphi \circ y_{n_1}\| \le \tfrac{1}{3}a,$$

$$\int_{S_2} \|\varphi \circ x_{n_2} - \varphi \circ y_{n_2}\| \ge \tfrac{2}{3}a, \qquad \mu(S_2) \le \delta_1 \le \tfrac{1}{2}\mu(S_1).$$

Suppose we have selected functions $\{x_{n_i}\}$, $\{y_{n_i}\}$ and measurable sets S_i, $1 \le i \le k$, as well as positive numbers δ_i, $1 \le i \le k - 1$, such that

(a) $\mu(S_i) \le \delta_{i-1} \le \tfrac{1}{2}\mu(S_{i-1})$,
(b) $E \in \Sigma$ and $\mu(E) \le 2\delta_i \Rightarrow \int_E \|\varphi \circ x_{n_i} - \varphi \circ y_{n_i}\| \, d\mu \le a/3$, and
(c) $\int_{S_i} \|\varphi \circ x_{n_i} - \varphi \circ y_{n_i}\| \, d\mu \ge 2a/3$.

Let now δ_k be a positive number such that for $E \in \Sigma$ if $\mu(E) \le 2\delta_k$ then $\int_E \|\varphi \circ x_{n_k} - \varphi \circ y_{n_k}\| \, d\mu \le a/3$. Let $n_{k+1} > n_k$ be so chosen that there exists a measurable set T_0, $\mu(X \sim T_0) \le \delta_k$, and for $\xi \in T_0$, $\|\varphi(t, \xi) - \varphi(t', \xi)\| \le a/3\mu(X)$ if $|t - t'| \le 1/n_{k+1}$ and $t, t' \in [-d, d]$. Let $X \sim T_0 = S_{k+1}$. It is verified that

$$\int_{S_{k+1}} \|\varphi \circ x_{n_{k+1}} - \varphi \circ y_{n_{k+1}}\| \, d\mu \ge 2a/3,$$

$$\int_{S_{k+1}} \|\varphi \circ x_{n_k} - \varphi \circ y_{n_k}\| \, d\mu \le a/3$$

and

$$\mu(S_{k+1}) \le \delta_k \le \tfrac{1}{2}\mu(S_k).$$

Thus there exist sequences of functions $\{x_{n_i}\}$, $\{y_{n_i}\}$, of corresponding measurable sets $\{S_i\}$, and of positive numbers δ_i such that the conditions listed as (a), (b) and (c) above hold for all $i \geqq 1$. Further it is verified that

$$\mu\left(\bigcup_{l > k+1} S_l\right) \leqq 2\mu(S_{k+1}) \leqq 2\delta_k.$$

Hence if $R_k = S_k \sim \bigcup_{l = k+1} S_l$ then $\{R_k\}_{k > 1}$ is a disjoint sequence of measurable sets and from (b) it follows that

(*)
$$\int_{R_k} \|\varphi \circ x_{n_k} - \varphi \circ y_{n_k}\|\, d\mu \geqq \frac{a}{3}.$$

Let $x = \sum_{k \geqq 1} x_{n_k} I_{R_k}$ and $y = \sum_{k \geqq 1} y_{n_k} I_{R_k}$. Since $\varphi(x) - \varphi(y)$ is in L_B^1, $\|\varphi(x) - \varphi(y)\|$ is μ-summable. However, the inequality (*) clearly contradicts the summability of the function $\|\varphi(x) - \varphi(y)\|$. Thus the theorem holds for $\mu(x) < \infty$.

Let more generally $\mu(X) = \infty$ and assume that there exist sequences as described in (A). We proceed to construct subsequences $\{x_{n_k}\}$, $\{y_{n_k}\}$ and a sequence of disjoint sets $\{R_k\}$ of finite measure such that $\int_{R_k} \|\varphi \circ x_{n_k} - \varphi \circ y_{n_k}\|\, d\mu > a/2$. Let R_1 be a set of finite measure such that $\int_{R_1} \|\varphi \circ x_1 - \varphi \circ y_1\|\, d\mu > a/2$. The existence of R_1 is assured by (A). Then the result for the case $\mu(X) < \infty$ implies that, for n_2 sufficiently large, $\int_{R_1} \|\varphi \circ x_{n_2} - \varphi \circ y_{n_2}\|\, d\mu < a/2$. Hence there exists a set $R_2 \subset X \sim R_1$, $0 \leqq \mu(R_2) < \infty$, such that $\int_{R_2} \|\varphi \circ x_{n_2} - \varphi \circ y_{n_2}\|\, d\mu > a/2$. Again since $\mu(R_1 \cup R_2) < \infty$ our earlier result implies that for n_3 sufficiently large $\int_{R_1 \cup R_2} \|\varphi \circ x_{n_3} - \varphi \circ y_{n_3}\|\, d\mu < a/2$. Hence there exists a set $R_3 \subset X \sim (R_1 \cup R_2)$ such that $\mu(R_3) < \infty$ and $\int_{R_3} \|\varphi \circ x_{n_3} - \varphi \circ y_{n_3}\|\, d\mu > a/2$. Proceeding we arrive at sequences of functions $\{x_{n_k}\}$, $\{y_{n_k}\}$ and disjoint sets $\{R_k\}_{k \geqq 1}$ as claimed. It is clear that if $x = \sum_{k \geqq 1} x_{n_k} I_{R_k}$ and $y = \sum_{k \geqq 1} y_{n_k} I_{R_k}$ then $\|\varphi \circ x - \varphi \circ y\|$ is not μ-summable, contradicting the fact that $x \to \varphi \circ x$ maps $L^\infty(\mu) \to L_B^1$. This completes the proof of Theorem 2.

The representation result established in the preceding theorems can be carried over to the Banach spaces $L^p(\mu)$, $1 \leqq p < \infty$, of equivalence classes of p-summable real valued measurable functions, and this is done in Theorem 3. The proof is similar in details to that of Theorem 2 in [6] and hence the details are not supplied. Note that since $p < \infty$, F_t is now defined only for sets of finite measure.

THEOREM 3. *Let $1 \leqq p < \infty$. Suppose that (X, Σ, μ) is a σ-finite measure space and $F: L^p(\mu) \to B$ is an additive function satisfying*

(1) *F is continuous on $L^p(\mu)$,*

(2) *for each real number t, F_t has almost locally compact average range,*

(3) *for each real number t and each $T \in \Sigma$ satisfying $\mu(T) < \infty$, $V(F_t)(T) < \infty$,*

(4) *for each $T \in \Sigma$ satisfying $\mu(T) < \infty$, $\{F_t\}_{t \in R}$ when restricted to subsets of T is locally uniformly continuous in variation.*

Then there exists a function $\varphi \in \text{Car } B$ such that the mapping $x \to \varphi \circ x$ takes $L^p(\mu)$ into L_B^1, and for $x \in L^p(\mu)$

(*)
$$F(x) = \int_X \varphi \circ x \, d\mu.$$

Moreover φ *can be taken to satisfy* (a) $\varphi(0, \cdot) = 0$ *a.e. and is then unique up to sets of the form* $R \times N$ *with* N *a null set in* X.

Conversely for every $\varphi \in \text{Car } B$ *such that* (a) *holds and the mapping* $x \to \varphi \circ x$ *takes* $L^p(\mu) \to L_B^1$, *the formula* (*) *defines a B-valued additive function* F *satisfying the conditions* (1), (2), (3) *and* (4).

2. C-differentiable additive set functions. Motivated in part by the classical Lebesgue theorem on the derivatives of additive set functions, several authors have studied the differentials of vector valued additive set functions. See for example Clarkson [1], Pettis [8] and Phillips [9]. For convenience of reference we state here a theorem in [9] after recalling pertinent definitions.

Let G be an additive function defined on the class Σ of Lebesgue measurable sets in $(0, 1)$ into a Banach space B. If $\xi \in (0, 1)$ then a vector $f(\xi) \in B$ is said to be the *derivative of* G *at* ξ if $\lim_{\mu(J) \to 0} \| G(J)/\mu(J) - f(\xi) \| = 0$ where J is an arbitrary interval containing ξ and μ is Lebesgue measure. If the derivative exists at almost all points $\xi \in (0, 1)$ then G is said to be (Lebesgue) differentiable and the function f which is defined a.e. is called the *derivative* (pseudo-derivative according to [9]) of G. If $G: R \times \Sigma \to B$ is a function such that $G(t, \cdot)$ is a differentiable additive function with f_t as the derivative then G is said to be *Carathéodory differentiable* (*C*-differentiable) if there exists a null set N such that for all $\xi \notin N$ the mapping $t \to f_t(\xi)$ is a continuous function on $R \to B$. We proceed to state Theorem 8.1 of [9], which characterizes differentiable set functions of finite variation, in a partial form suitable for our purpose.

THEOREM [PHILLIPS]. *A sufficient condition that an additive function* G *on measurable sets in* $(0, 1)$ *to a Banach space* B *be differentiable and of finite variation is that* G *be a Bochner integral.* G *is then the Bochner integral of its derivative.*

We proceed to state the next theorem after establishing a lemma. Throughout this section μ is Lebesgue measure on $(0, 1)$.

LEMMA 4. *Let* F *be an additive function on* $L^\infty(\mu)$ *into a Banach space* B. *Suppose* $x_n \to x$ *boundedly a.e. implies* $\| F(x_n) - F(x) \| \to 0$. *Then the following statements are equivalent*:

(a) *The function* F *satisfies conditions* (1), (2), (3) *and* (4) *in Theorem* 1.

(b) *If* $G: R \times \Sigma \to B$ *is the function* $G(t, E) = F(t I_E)$ *then* G *is C-differentiable.*

Proof. If F has property (a) then by Theorem 1 there exists a function $\varphi \in \text{Car } B$ such that $F(t I_E) = \int_E \varphi(t, \xi) \, d\mu(\xi)$. Hence by Phillips theorem, for each real number t, $G(t, \cdot)$ is differentiable at ξ, μ a.e. Since $\varphi \in \text{Car } B$ it follows that (a) $\Rightarrow$ (b). Next let us assume F satisfies the condition in (b). It follows that there exists a function $\varphi \in \text{Car } B$ such that for all simple functions x

$$(*) \qquad\qquad F(x) = \int_{(0, 1)} \varphi(x(\xi), \xi) \, d\mu(\xi).$$

Since F is continuous relative to bounded convergence a.e., proceeding as in the proof of Theorem 2 the equation (*) is verified for all $x \in L^\infty(\mu)$. But then it follows from Theorem 2 that F has all the properties stated in (a).

From the preceding lemma we deduce the following representation theorem.

THEOREM 4. *Let F be an additive function on $L^\infty(\mu) \to B$ satisfying the condition* (1) *in Theorem 1. Then F admits an integral representation with a kernel $\varphi \in \text{Car } B$ if and only if the function G in* (b) *of Lemma 4 is C-differentiable.*

3. An example. In this section we discuss an example to show that the representation Theorems 1 and 3 are false if we drop the hypothesis (4) on the additive function F. This also shows that in the preceding representation theorem we need the C-differentiability of the function G.

EXAMPLE. There exists a continuous additive function $F: L^\infty(\mu) \to H$, where the measure space is the unit interval with Lebesgue measure and H is a separable Hilbert space, such that F satisfies the conditions (1), (2) and (3) of Theorem 1 and is uniformly continuous on bounded subsets of $L^\infty(\mu)$. However, F does not admit an integral representation with a kernel $\varphi \in \text{Car } H$.

Let I_n denote the rectangle $(1/2^n, 1/2^{n-1}] \times (0, 1]$ in the cartesian plane for $n \geq 1$. Let $\{J_{nm}\}_{1 \leq m \leq 2^n}$ be a subdivision of I_n into 2^n rectangles where

$$J_{nm} = \left(\frac{1}{2^n}, \frac{1}{2^{n-1}}\right] \times \left(\frac{m-1}{2^n}, \frac{m}{2^n}\right], \qquad 1 \leq m \leq 2^n.$$

Next for each pair (n, m), $n \geq 1$, $1 \leq m \leq 2^n$, let $\mathcal{O}_{nm}$ be a continuous real valued function on $R \times (0, 1]$ satisfying the three conditions listed in (I) below:

(I)

    (a) support $\mathcal{O}_{nm} \subset$ interior J_{nm},

    (b) $\int \mathcal{O}_{nm}\left(\dfrac{2^{-n} + 2^{1-n}}{2}, \xi\right) d\mu(\xi) = \dfrac{1}{2^n}$,

    (c) $0 \leq \mathcal{O}_{nm}(t, \xi) \leq 2$.

Let $\{e_{nm}\}$, $n \geq 1$, $1 \leq m \leq 2^n$, be an orthonormal system in the Hilbert space H. Define $\varphi: R \times (0, 1] \to H$ as follows:

(II)
$$\varphi = \sum_{n, m} e_{nm} \mathcal{O}_{nm}.$$

Thus φ is continuous except at those points (h, ξ) with $h = 0$, and is identically zero on $\{(-\infty, 0] \cup [1, \infty)\} \times (0, 1]$.

We proceed to show that the function $F: L^\infty(\mu) \to H$ given by

(III)
$$F(x) = \int_{[0,1]} \varphi \circ x(\xi) \, d\mu(\xi)$$

where $\varphi \circ x(\xi) = \varphi(x(\xi), \xi)$ satisfies conditions (1), (2), (3) of Theorem 1 but not (4).

Since it is readily verified that F is additive we proceed to show F satisfies condition (1). We note first that the function $\varphi \circ x$ is Bochner measurable whenever x

is a measurable real valued function. This follows by the same argument used for φ's in Car H, once one restricts attention to the support of x. Moreover since, by (I)(c), $\|\varphi \circ x(\xi)\| \leqq 2$ it follows that $\varphi \circ x$ is Bochner integrable. Now for each $\varepsilon > 0$ there exists an integer N such that

$$\left\| \int_{[0,1]} \varphi \circ x \, d\mu \right\| < \varepsilon/3 \quad \text{whenever} \quad \|x\|_\infty \leqq 1/2^N.$$

For according to (II)

$$\int \varphi \circ x \, d\mu = \sum_{n \geqq N+1; \, 1 \leqq m \leqq 2^n} C_{nm} e_{nm}, \quad C_{nm} \geqq 0,$$

where

$$C_{nm} = \int_{[0,1]} \mathcal{O}_{nm}(x(\xi), \xi) \, d\mu(\xi) \leqq 2\mu(E_{nm})$$

and

$$E_{nm} = \{\xi \mid (x(\xi), \xi) \in J_{nm}\}$$
$$= \{\xi \mid x(\xi) \in (1/2^n, 1/2^{n+1}]\} \cap ((m-1)/2^n, m/2^n].$$

Since the E_{nm} are disjoint and $\bigcup E_{nm} \subset (0, 1]$ it follows that

$$\sum_{n \geqq N+1; \, 1 \leqq m \leqq 2^n} C_{nm} \leqq 2\mu((0, 1]) = 2.$$

Moreover by (I)(c)

$$C_{nm} \leqq 2\mu(((m-1)/2^n, m/2^n]) \leqq 2^{-N},$$

since $n \geqq N+1$. Thus

$$\left\| \int_{[0,1]} \varphi \circ x \, d\mu \right\|^2 = \sum_{n \geqq N+1; \, 1 \leqq m \leqq 2^n} C_{nm}^2 \leqq 2^{-N} \sum C_{nm} \leqq 2^{1-N},$$

which in turn is less than $\varepsilon/3$ for sufficiently large N.

Suppose now that the sequence $\{x_n\}$ in $L^\infty(\mu)$ converges boundedly a.e. to $x \in L^\infty(\mu)$. Since $\varphi(h, \xi) = 0$ for $h < 0$, it follows that $F(x) = F(x^+)$, $F(x_n) = F(x_n^+)$, and hence we may assume that x and $\{x_n\}$ are nonnegative. With N as in the preceding paragraph, select a real number h_0, $2^{-N-1} < h_0 < 2^{-N}$, such that $x^{-1}(\{h_0\})$ is a null set. Then with x', x'', x_n', x_n'' defined by

$$x' = x I_{x^{-1}([h_0, \infty))}, \qquad x'' = x I_{x^{-1}([0, h_0))}$$
$$x_n' = x_n I_{x_n^{-1}([h_0, \infty))}, \qquad x_n'' = x_n I_{x_n^{-1}([0, h_0))},$$

the following relations hold: $x_n' \to x'$ boundedly a.e., $\|x_n''\|_\infty, \|x''\|_\infty \leqq 2^{-N}$. Thus

$$\|F(x_n) - F(x)\| \leqq \|F(x_n') - F(x')\| + \|F(x_n'')\| + \|F(x'')\|$$
$$\leqq 2\varepsilon/3 + \|F(x_n') - F(x')\|,$$

by definition of N. Noting that φ restricted to $[2^{-N-1}, \infty) \times [0, 1]$ coincides there with a Carathéodory function, we deduce from Theorem 2 that $\|F(x'_n) - F(x')\| \to 0$. Thus for sufficiently large n, $\|F(x_n) - F(x)\| < \varepsilon$, which shows that condition (1) holds.

Since for each real number t the set function F_t is a Bochner integral, it follows by the main theorem in [10] that conditions (2) and (3) hold.

However, F fails to satisfy condition (4). For from (I)(b) it follows that if $h_n = \frac{1}{2}(2^{-n} + 2^{1-n})$ then $\int \|\varphi(h_n, \xi) - \varphi(0, \xi)\| \, d\mu(\xi) = \int \|\varphi(h_n, \xi)\| \, d\mu(\xi) = 2^n \cdot 1/2^n = 1$, although $h_n \to 0$.

Let now M be a positive real number. We wish to show that F is uniformly continuous on the ball $S_M = \{x \mid x \in L^\infty(\mu), \|x\|_\infty \leq M\}$. If F is not uniformly continuous on S_M then there exist two sequences $\{x_n\}$, $\{y_n\}$ with $\|x_n - y_n\|_\infty \to 0$ and $\|F(x_n) - F(y_n)\| > \varepsilon$ for some positive number ε independent of n. As above, we can assume that x_n, y_n are nonnegative. Select N as in the preceding paragraphs. We decompose each of the functions x_n, $n \geq 1$, as $x_n = x'_n + x''_n$ with x'_n, x''_n having disjoint supports A'_n and A''_n. Here $\|x''_n\|_\infty \leq \frac{1}{2} 2^{-N}$ and $x'_n(\xi) > \frac{1}{2} 2^{-N}$ for all ξ in its support A'_n. Define $y'_n = y_n I_{A'_n}$, $y''_n = y_n - y'_n$. Then for n sufficiently large $0 \leq y''_n < 2^{-N}$ and $y'_n \geq \frac{1}{4} 2^{-N}$ on A'_n. Now consider

$$\|F(x_n) - F(y_n)\| \leq \|F(x'_n) - F(y'_n)\| + \|F(x''_n)\| + \|F(y''_n)\|$$
$$\leq 2\varepsilon/3 + \|F(x'_n) - F(y'_n)\|$$

by definition of N. Let us examine the last term. Now $x'_n, y'_n \geq \delta = \frac{1}{4} 2^{-N}$ on A'_n. Noting that φ restricted to $[\delta, \infty) \times (0, 1]$ coincides there with a Carathéodory function we deduce from Theorem 2 that $\|F(x'_n) - F(y'_n)\| \to 0$, leading to a contradiction.

Since F is uniformly continuous on the sets S_M and yet does not satisfy condition (4), we see that the converse of Lemma 1 fails (see Remark 1(a) in §1).

4. Representation of additive functions on $L^p_E \to B$. In the present section we give generalized forms of the preceding representation theorems, namely the theorems which occur on replacing the real line R by an infinite-dimensional separable Banach space E.

Let (X, Σ, μ) be a complete σ-finite positive measure space and E, B be two Banach spaces. As already noted L^p_E $(1 \leq p < \infty)$ is the Banach space of equivalence classes of E-valued Bochner measurable functions f on X such that the real valued function $x \to \|f(x)\|^p$ is μ-summable. Similarly L^∞_E is the Banach space of equivalence classes of E-valued Bochner measurable functions f with the norm defined by $\|f\| = \text{ess sup}_{x \in X} \|f(x)\|$. A function $\varphi \colon E \times X \to B$ is said to be in the *uniform Carathéodory class* relative to E on $X \to B$, in brief $\varphi \in U - \text{Car}\, (E, B)$, if $\varphi(e, \cdot)$ is a B-valued Bochner measurable function for each vector $e \in E$ and $\varphi(\cdot, \xi)$ is uniformly continuous on bounded subsets of E for all ξ outside a μ-null set. Given a p, $1 \leq p \leq \infty$, $\varphi \in U - \text{Car}\, (E, B)$ is said to be in $U - \text{Car}^p\, (E, B)$ if the composition operator $x \to \varphi \circ x$ where $\varphi \circ x(\xi) = \varphi(x(\xi), \xi)$ maps L^p_E into L^1_B.

Before proceeding to the main theorem of this section let us note that the following properties of the real line R were used in the proofs of the theorems in §1. (1) The separability of the real line was used in the proof of Lemma 3 preceding Theorem 1. (2) The local compactness of the real line was involved in the proof of Theorem 2 in order to conclude that the functions $\varphi(\cdot, \xi): R \to B$ are uniformly continuous on bounded subsets of the real line. With the aid of these observations it is easily verified that the theorems stated in §1 remain valid (exactly as formulated) when R is replaced by any *finite-dimensional* Banach space E[1]. On the other hand, for the infinite-dimensional case we have the following result. [In analogy with the terminology in §1 we let F_e denote the set function defined on $\Sigma \to B$ (or only on sets in Σ of finite measure) by the equation $F_e(M) = F(eI_M)$.]

THEOREM 5. *Let (X, Σ, μ) be as in Theorem 1, let E be a separable Banach space, and let B be an arbitrary Banach space. Let $F: L_E^p \to B$ $(1 \leq p \leq \infty)$ be an additive function satisfying the following conditions*:

1. *For each vector $e \in E$ the set function F_e which carries sets in Σ of finite measure to B has almost locally compact average range.*

2. *For each $e \in E$, if $T \in \Sigma$, $\mu(T) < \infty$, then $V(F_e)(T) < \infty$.*

3. *On each set T of finite measure, the family of set functions $\{F_e\}_{e \in E}$ is locally uniformly continuous in variation.*

4. *F is continuous relative to L_E^p norm, if $p < \infty$, and is continuous with respect to bounded a.e. convergence if $p = \infty$.*

Then there exists a function $\varphi \in U - \mathrm{Car}^p (E, B)$ such that

$$\text{(*)} \qquad\qquad F(x) = \int_X \varphi \circ x \, d\mu \quad \text{for } x \in L_E^p.$$

Moreover φ can be taken to satisfy

$$\text{(a)} \qquad\qquad \varphi(0, \cdot) = 0 \quad a.e.$$

and is then unique up to sets of the form $E \times N$ with N a null set in X.

Conversely, for any $\varphi \in U - \mathrm{Car}^p (E, B)$ satisfying (a), (*) defines an additive function from $L_E^p \to B$ satisfying conditions 1, 2, 3 and 4.

The proof of Theorem 5 is very similar to the proofs of Theorems 1, 2 and 3 and is easily supplied with the aid of observations (1) and (2) in the paragraph preceding the statement of the present theorem. Hence we omit the details.

As noted earlier, when $B = R$ the following result for linear functionals appears as a corollary. Here E^* denotes the dual of E, and a function $f: X \to E^*$ is said to be *weakly measurable* provided that (a) for some null set N the w^*-closure of $f(X \sim N)$ is separable in the w^*-topology, and (b) $f^{-1}(G)$ is measurable for each w^*-open set $G \subset E^*$.

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[1] Likewise for the theorems in [6] and [2].

THEOREM 6 (DIEUDONNÉ [13], [14], EDWARDS [4, THEOREM 8.20.3]). *Suppose that E is a separable Banach space and F is a continuous linear functional on L_E^p, where $1 \leq p < \infty$. Then there exists a function f mapping X into E^* that is weakly measurable and such that*

$$(\alpha) \qquad F(x) = \int \langle f, x \rangle \, d\mu \quad \text{for each } x \in L_E^p,$$

$$(\beta) \qquad \|F\| = \left[\int \|f\|^{p'} \, d\mu \right]^{1/p'}, \quad \text{where } 1/p' + 1/p = 1, \quad p > 1,$$

$$= \text{ess sup } \|f(\cdot)\|, \qquad p = 1.$$

Conversely any weakly measurable function f for which the right-hand side of (β) is finite defines, via (α), a continuous linear functional F on L_E^p for which (β) holds.

In order to obtain the proof we develop certain preliminary results.

LEMMA 5. *If F is a continuous linear functional on L_E^p, where $1 \leq p < \infty$, then F satisfies conditions 1, 2, 3, 4 of Theorem 5.*

Proof. Since F is linear it is verified that F is an additive function on $L_E^p \to R$. Thus for each vector $e \in E$, F_e is an additive set function on sets $T \in \Sigma$ such that $\mu(T) < \infty$. Let $\Sigma_T = \{T \cap M \mid M \in \Sigma\}$. Since F is continuous and $F(0) = 0$ it is verified that $F_e \mid \Sigma_T$ is μ-absolutely continuous. Hence by the (ordinary) Radon-Nikodym Theorem and the main theorem in Rieffel [10] (for real measures) we infer that F has properties 1 and 2. Condition 4 holds by hypothesis.

We proceed to verify condition 3. Let $T \in \Sigma$, $\mu(T) < \infty$, and $\varepsilon > 0$ be given. Let $\delta = \varepsilon / \|F\| \mu(T)^{1/p} (1 + \varepsilon)$. Now suppose $\{T_i\}_{1 \leq i \leq n}$ is a measurable partition of T and $\{e_i\}_{1 \leq i \leq n}$, $\{e_i'\}_{1 \leq i \leq n}$ in E satisfy $\|e_i - e_i'\| \leq \delta$. Then by the definition of the variation, there exist partitions $\{T_{ij}\}_{1 \leq j \leq m_i}$ of the $\{T_i\}$ such that

$$(1) \qquad \sum_i V(F_{e_i} - F_{e_i'})(T_i) \leq (1 + \varepsilon) \sum_i \sum_j |F_{e_i}(T_{ij}) - F_{e_i'}(T_{ij})|$$

$$= (1 + \varepsilon) \sum_{i,j} |F_{e_i - e_i'}(T_{ij})|.$$

Let

$$P = \{(i, j) \mid 1 \leq i \leq n, 1 \leq j \leq m_i \text{ and } F_{e_i - e_i'}(T_{ij}) \geq 0\},$$

$$Q = \{(i, j) \mid 1 \leq i \leq n, 1 \leq j \leq m_i \text{ and } F_{e_i - e_i'}(T_{ij}) < 0\}.$$

Then

$$(2) \qquad \sum_{i,j} |F_{e_i - e_i'}(T_{ij})| = \sum_{(i,j) \in P} F_{e_i - e_i'}(T_{ij}) - \sum_{(i,j) \in Q} F_{e_i - e_i'}(T_{ij})$$

$$= F \left(\sum_{(i,j) \in P} (e_i - e_i') I_{T_{ij}} - \sum_{(i,j) \in Q} (e_i - e_i') I_{T_{ij}} \right)$$

$$\leq \|F\| \delta \left(\sum \mu(T_{ij}) \right)^{1/p}.$$

It follows from (1) and (2) that

$$\sum V(F_{e_i - e_j})(T_i) \leq (1+\varepsilon)\|F\|(\mu(T))^{1/p}\delta < \varepsilon.$$

Hence F satisfies condition 3.

LEMMA 6. *If for some $\varphi \in U - \mathrm{Car}^p (E, R)$ the functional F defined on L_E^p by*

$$F(x) = \int_X \varphi(x(\xi), \xi) \, d\mu(\xi)$$

is linear, then there exists a μ-null set N and a w^*-measurable function $f \colon X \to E^*$, with $\|f(\cdot)\|$ μ-measurable, such that $\varphi(e, \xi) = f(\xi)(e)$ for $e \in E$, $\xi \notin N$.

Proof. Since F is linear it is verified that if $A \in \Sigma$, $\mu(A) < \infty$, then for any two vectors $e, e' \in E$,

$$\int_A \varphi(e+e', \xi) \, d\mu(\xi) = F((e+e')I_A) = F(eI_A) + F(e'I_A)$$

$$= \int_A \varphi(e, \xi) \, d\mu(\xi) + \int_A \varphi(e', \xi) \, d\mu(\xi).$$

Hence by the uniqueness of Radon-Nikodym derivatives it follows that $\varphi(e+e', \xi) = \varphi(e, \xi) + \varphi(e', \xi)$ a.e. Since E is separable and $\varphi \in U - \mathrm{Car}(E, B)$ it is further true that there exists a μ-null set N_1 such that for all $\xi \notin N_1$, and for any pair $e, e' \in E$, $\varphi(e+e', \xi) = \varphi(e, \xi) + \varphi(e', \xi)$. It follows from this that, for $\xi \notin N_1$, $\varphi(ce, \xi) = c\varphi(e, \xi)$ whenever c is rational. Hence, since the rationals are dense in R and $\varphi \in U - \mathrm{Car}(E, B)$ one has that for all $e \in E$ and all $c \in R$, $\varphi(ce, \xi) = c\varphi(e, \xi)$ for ξ outside a null set $N \supset N_1$. Moreover, without loss of generality we can assume N to be such that $\varphi(\cdot, \xi)$ is uniformly continuous for $\xi \notin N$. Thus if $f(\xi)$ is the function on $E \to R$ defined by $f(\xi)(e) = \varphi(e, \xi)$ then $f(\xi) \in E^*$ for all $\xi \notin N$ and, by redefining φ so that $\varphi(E \times N) = 0$, we can assume that $f(\xi) \in E^*$ for all $\xi \in X$. Further for each $e \in E$, $f(\cdot)(e) = \varphi(e, \cdot)$ is a real valued measurable function on X. Thus since E is separable it follows from Proposition 8.15.3 of [4] that $f \colon X \to E^*$ is a w^*-measurable function and that the function $\xi \to \|f(\xi)\|$ is a real valued measurable function.

Proof of the theorem. By Lemma 5, F satisfies the hypotheses of Theorem 5. Hence F has a representation of the form (∗) with $\varphi \in U - \mathrm{Car}^p (E, R)$. By Lemma 6 this yields the formula

$$F(x) = \int f(\xi)(x(\xi)) \, d\mu(\xi)$$

with $f \colon X \to E^*$ satisfying the measurability conditions stated in the lemma.

We now proceed to verify equation (β). Let U_n be the closed ball in E^* of radius n $(n \geq 1)$. Let C_n be the measurable set $f^{-1}(U_n)$. Then $\bigcup_{n \geq 1} C_n = X$ and $f|C_n$ is a w^*-pointwise almost uniform limit of a sequence of countably valued functions

$\{l_n^i\}_{i \geqq 1}$ with support $(l_n^i) \subset C_n$. This follows from the metrizability of U_n in the w^*-topology since by its measurability f is almost separably valued. Utilizing this fact it is easy to show that there exists for each positive ε a countably valued function $x^n \colon X \to E$ with support in C_n and unit L_E^p-norm such that

$$\|F\| \geqq F(x^n) \geqq \left[\int_{C_n} \|f\|^{p'} \right]^{1/p'} (1-\varepsilon), \qquad p > 1,$$

$$\geqq \operatorname*{ess\,sup}_{C_n} \|f(\cdot)\|(1-\varepsilon), \qquad p = 1.$$

It follows from this that $\|f\| \in L_R^{p'}$ and $\|F\|^{p'} = \int_X \|f\|^{p'} \, d\mu$ (for $p=1$, $\|F\|$ $= \operatorname{ess\,sup} \|f(\cdot)\|$).

For the converse note that if $f \colon X \to E^*$ is a w^*-measurable function and $\int_X \|f\|^{p'} \, d\mu < \infty$ (for $p=1$, $\operatorname{ess\,sup} \|f(\cdot)\| < \infty$) then the function $\varphi \colon E \times X \to R$ defined by $\varphi(e, \xi) = \langle f(\xi), e \rangle$ is in $U - \operatorname{Car}(E, R)$. The verification that $\varphi \in U$ $-\operatorname{Car}^p(E, R)$ and that the linear functional $F \colon L_E^p \to R$ defined by $F(x)$ $= \int \langle f(\xi)(x(\xi)) \rangle \, d\mu(\xi)$ is continuous and satisfies (β) follows by the methods of the preceding paragraph.

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